

# Modal identification of structures from roving input data by means of maximum likelihood estimation of the state space model

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## Abstract

The usual way to perform a forced vibration test is to fix accelerometers to the structure, then apply a known input and record the response of the structure. An interesting variant consists of subsequently repeating the test but with the input force applied to different points, roving the input from test to test but not the sensors. In a second phase, individual frequency response functions are combined in a single frequency response function which takes into account all the information. The advantage of this procedure is double: first, the mode shapes can be estimated at all points where inputs are applied (using the dynamic reciprocity theorem); and second, the measured forces allow to scale the mode shapes to unit modal mass. This problem is solved in technical literature in the frequency domain. In this work we propose a state space model that can be used to solve the problem in the time domain. We also propose a maximum likelihood algorithm to estimate such a model.

## 1 Introduction

A roving sensor modal test consists on estimating the modal parameters using the data recorded in different tests changing the position of the sensors from one test to the following: some sensors stay at the same position from test to test, and other sensors change the position until all the tested points are covered. The known input, if applied, must be applied at the same point in all the test setups. This technique has reached great popularity in the civil engineering field because of the size of the civil structures and because the tests can be performed in operational conditions. The modal parameters can be extracted in the frequency domain (for example, using the Frequency Domain Decomposition method) and in the time domain (for example, with the state space model).

Another way to run the test is to place sensors at fixed locations and change the loading point from one test to the next covering all the desired points (roving excitation modal test). This technique is quite popular in experimental modal analysis with impact excitations (hammer) for several reasons: structures are usually small and roving the sensors can affect to the modal parameters; it is easier to change the position of the hammer impact than roving the sensors; it is a less expensive approach seen from a hardware point of view. The modal parameters are extracted in the bibliography using the dynamic reciprocity theorem (or Betti-Rayleigh theorem) in the frequency domain [1]. However, there are no examples in the time domain. The purpose of this work is to propose a method to estimate the modal parameters in the time domain: first, we present a state space model to deal with data recorded in a roving excitation test; and second, we propose a maximum likelihood algorithm to estimate that model (the Expectation-Maximization algorithm).

The performance of the proposed method is analyzed using a simulated structure and also using vibration data recorded in a real structure.

## 2 State space model for roving input data and its estimation using maximum likelihood

### 2.1 State space model for roving input data

Let be a linear, time invariant mechanical/structural system. Let us consider  $n_r$  sample records with  $N$  measurements each,

$$Y_N^{(r)} = \{y_1^{(r)}, y_2^{(r)}, \dots, y_N^{(r)}\}, \quad r = 1, 2, \dots, n_r \quad (1)$$

$$U_N^{(r)} = \{u_1^{(r)}, u_2^{(r)}, \dots, u_N^{(r)}\}, \quad r = 1, 2, \dots, n_r \quad (2)$$

where  $y_t^{(r)} \in \mathbb{R}^{n_o}$  is the measured output vector for record  $r$ , and  $u_t^{(r)} \in \mathbb{R}^{n_i}$  is the measured input vector for record  $r$ . Assume the inputs vary from record to record (point of application, number of inputs, ...) but the output sensors remain fixed. The state space model we propose to use for these data is

$$x_{t+1}^{(r)} = Ax_t^{(r)} + B^{(r)}u_t^{(r)} + w_t^{(r)}, \quad w_t^{(r)} \sim N(0, Q) \quad (3)$$

$$y_t^{(r)} = Cx_t^{(r)} + D^{(r)}u_t^{(r)} + v_t^{(r)}, \quad v_t^{(r)} \sim N(0, R) \quad (4)$$

where  $t$  denotes the time instant, of a total number  $N$ , measured with constant sampling time  $\Delta t$ ;  $x_t^{(r)} \in \mathbb{R}^{n_s}$  is the state vector for record  $r$ ;  $n_o$ ,  $n_i$  and  $n_s$  are the number of outputs, inputs and the size of the state vector, respectively;  $A \in \mathbb{R}^{n_s \times n_s}$  is the transition state matrix describing the dynamics of the system;  $B^{(r)} \in \mathbb{R}^{n_s \times n_{ir}}$  is the input matrix ( $n_{ir}$  is the number of inputs at record  $r$ );  $C \in \mathbb{R}^{n_o \times n_s}$  is the output matrix, which is describing how the internal state is transferred to the the output measurements  $y_t^{(r)}$ ;  $D^{(r)} \in \mathbb{R}^{n_o \times n_{ir}}$  is the direct transmission matrix. The noise vectors comprise unmeasurable signals:  $w_t^{(r)} \in \mathbb{R}^{n_s}$  is the process noise due to disturbances and modelling discrepancies, while  $v_t^{(r)} \in \mathbb{R}^{n_o}$  is the measurement noise due to sensor inaccuracy. Both are assumed to be zero-mean, white noise sequences with covariance matrices  $Q$  and  $R$ , respectively.

It is important to note that matrix  $A$  is invariant because the system is time invariant (it is the same from record to record), and matrix  $C$  is also invariant because the sensors location do not change. On the other hand,  $B^{(r)}$  and  $D^{(r)}$  are record-depending because the system input are different from record to record. A more practical way of writing this model is

$$x_{t+1}^{(r)} = Ax_t^{(r)} + BL^{(r)}u_t^{(r)} + w_t^{(r)}, \quad w_t^{(r)} \sim N(0, Q) \quad (5)$$

$$y_t^{(r)} = Cx_t^{(r)} + DL^{(r)}u_t^{(r)} + v_t^{(r)}, \quad v_t^{(r)} \sim N(0, R) \quad (6)$$

where  $L^{(r)} \in \mathbb{R}^{n_s \times n_{ir}}$  is a selection matrix formed by ones and zeros verifying  $B^{(r)} = L^{(r)}B$  and  $D^{(r)} = L^{(r)}D$ . For example, consider a system with  $n_s = 4$ ,  $n_i = 3$  and  $n_o = 1$ . We can write

$$Bu_t = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \\ b_{41} & b_{42} & b_{43} \end{bmatrix} \begin{bmatrix} u_{1,t} \\ u_{2,t} \\ u_{3,t} \end{bmatrix}, \quad Du_t = [d_{11} \quad d_{12} \quad d_{13}] \begin{bmatrix} u_{1,t} \\ u_{2,t} \\ u_{3,t} \end{bmatrix}.$$

If we measure now only the input  $u_{1,t}$  we can write

$$B^{(1)}u_{1,t} = \begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \\ b_{41} \end{bmatrix} u_{1,t}, \quad D^{(1)}u_{1,t} = d_{11}u_{1,t}.$$

Using the selection matrices:

$$B^{(1)}u_{1,t} = \begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \\ b_{41} \end{bmatrix} u_{1,t} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \\ b_{41} & b_{42} & b_{43} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u_{1,t} = BL^{(1)}u_{1,t}.$$

$$D^{(1)}u_{1,t} = d_{11}u_{1,t} = \begin{bmatrix} d_{11} & d_{12} & d_{13} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u_{1,t} = DL^{(1)}u_{1,t}.$$

Therefore,  $L^{(1)} = [1 \ 0 \ 0]^T$ . The location matrices are known for each setup  $r$ .

The unknown parameters of model (5)-(6) are

$$\theta = \{A, B, C, D, Q, R, \bar{x}_0^{(r)}, P_0^{(r)}\}, \quad r = 1, 2, \dots, n_r \quad (7)$$

where  $\bar{x}_0^{(r)}$  and  $P_0^{(r)}$  are the mean and variance of the initial state  $x_0^{(r)}$  respectively (which is assumed to be normal distributed).

## 2.2 Maximum likelihood estimation: EM algorithm

We assume all the outputs  $Y_N = \{Y_N^{(1)}, Y_N^{(2)}, \dots, Y_N^{(n_r)}\}$ , inputs  $U_N = \{U_N^{(1)}, U_N^{(2)}, \dots, U_N^{(n_r)}\}$  and states  $X_{N+1} = \{X_{N+1}^{(1)}, X_{N+1}^{(2)}, \dots, X_{N+1}^{(n_r)}\}$  are known (note that  $Y_N^{(r)} = \{y_1^{(r)}, y_2^{(r)}, \dots, y_N^{(r)}\}$ ,  $U_N^{(r)} = \{u_1^{(r)}, u_2^{(r)}, \dots, u_N^{(r)}\}$  and  $X_{N+1}^{(r)} = \{x_1^{(r)}, x_2^{(r)}, \dots, x_{N+1}^{(r)}\}$  are the outputs, inputs and states for one individual record  $r$ , respectively). The density function for record  $r$  is given by (see [2])

$$f_\theta^{(r)}(X_{N+1}^{(r)}, Y_N^{(r)} | U_N^{(r)}) = f_0^{(r)}(x_0^{(r)}) \prod_{t=1}^N f_1^{(r)}(X_{t+1}^{(r)} | X_t^{(r)}, U_t^{(r)}) \prod_{t=1}^N f_2^{(r)}(Y_t^{(r)} | X_t^{(r)}, U_t^{(r)}), \quad (8)$$

where under Gaussian assumption:

$$\begin{aligned} f_0^{(r)}(x_0^{(r)}) &= \frac{1}{(2\pi)^{n_s/2} |P_0^{(r)}|^{1/2}} \exp \left( -\frac{1}{2} (x_0^{(r)} - \bar{x}_0^{(r)})^T (P_0^{(r)})^{-1} (x_0^{(r)} - \bar{x}_0^{(r)}) \right), \\ f_1^{(r)}(X_{t+1}^{(r)} | X_t^{(r)}, U_t^{(r)}) &= \\ &= \frac{1}{(2\pi)^{n_s/2} |Q|^{1/2}} \exp \left( -\frac{1}{2} (x_{t+1}^{(r)} - Ax_t^{(r)} - BL^{(r)}u_t^{(r)})^T Q^{-1} (x_{t+1}^{(r)} - Ax_t^{(r)} - BL^{(r)}u_t^{(r)}) \right), \\ f_2^{(r)}(Y_t^{(r)} | X_t^{(r)}, U_t^{(r)}) &= \\ &= \frac{1}{(2\pi)^{n_o/2} |R|^{1/2}} \exp \left( -\frac{1}{2} (y_t^{(r)} - Cx_t^{(r)} - DL^{(r)}u_t^{(r)})^T R^{-1} (y_t^{(r)} - Cx_t^{(r)} - DL^{(r)}u_t^{(r)}) \right). \end{aligned}$$

Thus, if we consider  $n_r$  independent registers, the joint density function  $f_\theta(X_{N+1}, Y_N | U_N)$  will be the product of the individual ones:

$$f_\theta(X_{N+1}, Y_N | U_N) = \prod_{r=1}^{n_r} f_\theta^{(r)}(X_{N+1}^{(r)}, Y_N^{(r)} | U_N^{(r)}). \quad (9)$$

The complete data likelihood is defined by  $L(\theta) = f_\theta(X_{N+1}, Y_N | U_N)$ . In practice we generally work with the log-likelihood, so information is combined by addition and it can be written as a sum of the log-likelihood of each individual record:

$$\log L(\theta) = \sum_{r=1}^{n_r} \log f_\theta^{(r)}(X_{N+1}^{(r)}, Y_N^{(r)} | U_N^{(r)}) = \sum_{r=1}^{n_r} l^{(r)}(\theta), \quad (10)$$

where  $l^{(r)}(\theta)$  is the log-likelihood of record  $r$ , that is,  $l^{(r)}(\theta) = \log f_\theta^{(r)}(X_{N+1}^{(r)}, Y_N^{(r)} | U_N^{(r)})$ . It can be written as the sum of three uncoupled functions:

$$l^{(r)}(\theta) = -\frac{1}{2} [l_1^{(r)}(\bar{x}_0^{(r)}, P_0^{(r)}) + l_2^{(r)}(A, B, Q) + l_3^{(r)}(C, D, Q)], \quad (11)$$

where, ignoring constants:

$$l_1^{(r)}(\bar{x}_0^{(r)}, P_0^{(r)}) = \log |P_0^{(r)}| + (x_0^{(r)} - \bar{x}_0^{(r)})^T (P_0^{(r)})^{-1} (x_0^{(r)} - \bar{x}_0^{(r)}), \quad (12)$$

$$l_2^{(r)}(A, B, Q) = N \log |Q| + \sum_{t=1}^N (x_{t+1}^{(r)} - Ax_t^{(r)} - BL^{(r)}u_t^{(r)})^T Q^{-1} (x_{t+1}^{(r)} - Ax_t^{(r)} - BL^{(r)}u_t^{(r)}), \quad (13)$$

$$l_3^{(r)}(C, D, R) = N \log |R| + \sum_{t=1}^N (y_t^{(r)} - Cx_t^{(r)} - DL^{(r)}u_t^{(r)})^T R^{-1} (y_t^{(r)} - Cx_t^{(r)} - DL^{(r)}u_t^{(r)}). \quad (14)$$

The objective now is to maximize the log-likelihood given by equation (10). In this work we propose to use a maximization procedure based on the Expectation Maximization algorithm (see [2]). The EM algorithm is simple to apply since at each iteration the optimal solution for the unknown parameters can be obtained from explicit formulas. It consists on two steps: the E-step and the M-step.

- **E-step (expectation step):**

Given the measured outputs in all the records  $Y_N$ , the measured inputs in all the records  $U_N$  and a value for the parameters  $\theta_0$ , the log-likelihood (10) cannot be computed because the states  $X_{N+1}$  are unknown (in fact, the states are unobserved quantities). The method proposes to replace them with their expected values:

$$\begin{aligned} E[\log L(\theta) | Y_N, U_N, \theta_0] &= \sum_{r=1}^{n_r} E[l^{(r)}(\theta) | Y_N^{(r)}, U_N^{(r)}, \theta_0] = \sum_{r=1}^{n_r} E[l_1^{(r)}(\bar{x}_0^{(r)}, P_0^{(r)}) | Y_N^{(r)}, U_N^{(r)}, \theta_0] \\ &+ \sum_{r=1}^{n_r} E[l_2^{(r)}(A, B, Q) | Y_N^{(r)}, U_N^{(r)}, \theta_0] + \sum_{r=1}^{n_r} E[l_3^{(r)}(C, D, R) | Y_N^{(r)}, U_N^{(r)}, \theta_0^{(r)}]. \end{aligned} \quad (15)$$

- **M-step (maximization step):**

Maximizing  $E[\log L(\theta) | Y_N, U_N, \theta_0]$  with respect to the parameters  $\theta$ , constitutes the M-step. This is the strong point of the EM algorithm because the maximum values, obtained equating to zero the corresponding derivatives, are obtained from explicit formulas:

$$\frac{\partial}{\partial \theta} E[\log L(\theta) | Y_N, U_N, \theta_0] = 0 \Rightarrow \boxed{\theta = \hat{\theta}} \quad (16)$$

A new E-step can be now performed considering  $\theta_0 = \hat{\theta}$  what gives, applying the M-step, to a new value for the parameters,  $\hat{\theta}$ . This leads to an iterative procedure in which the two steps, expectation and maximization, are repeated until the likelihood is maximized. This procedure is called the EM algorithm.

### 3 Computation of modal parameters

Natural frequencies, modal damping ratios, mode shapes and modal masses can be retrieved from matrices  $A$ ,  $B$  and  $C$  using the following equations (see [3] for a general overview):

- The eigenvalues of  $A$  come in complex conjugate pairs and each pair represents one physical vibration mode. Assuming proportional damping, the  $j$ th eigenvalue of  $A$  has the form

$$\lambda_j = \exp \left( \left( -\zeta_j \omega_j \pm i \omega_j \sqrt{1 - \zeta_j^2} \right) \Delta t \right) \quad (17)$$

where  $\omega_j$  are the natural frequencies,  $\zeta_j$  are damping ratios, and  $\Delta t$  is the time step. Natural frequencies  $\omega_j$  and the damping ratios  $\zeta_j$  are then given by

$$\omega_j = \frac{|\ln(\lambda_j)|}{\Delta t} \quad (18)$$

$$\zeta_j = \frac{-\text{Real}[\ln(\lambda_j)]}{\omega_j \Delta t} \quad (19)$$

- The  $j$ th mode shape  $\phi_j \in \mathbb{R}^{n_o}$  evaluated at sensor locations can be obtained using the following expression:

$$\phi_{y,j} = C \psi_j \quad (20)$$

where  $\psi_j$  is the complex eigenvector of  $A$  corresponding to the eigenvalue  $\lambda_j$ .

- Finally, the modal mass corresponding to  $\phi_{y,j}$  can be computed using

$$m_{m,j} = \frac{e^{\lambda_j \Delta t} - 1}{\lambda_j (\lambda_j - \lambda_j^*)} \phi_{u,j}^T \gamma_j^\dagger \quad (21)$$

where  $\square^\dagger$  stands for the pseudo-inverse,  $\gamma_j \in \mathbb{R}^{n_i \times 1}$  stands for the  $j$ -th row of matrix  $\Gamma = V^{-1}B$ , with  $V$  meaning the eigenvectors matrix of  $A$ ;  $\phi_{u,j}$  is the modal vector  $\phi_{y,j}$  selected at the DOFs where the measured inputs are applied (considered as a column vector). Note that Equation (21) can be only used if the measured loads are applied to DOFs with sensors.

## 4 Numerical examples

### 4.1 Simulated data

We have considered a simply supported beam with the following characteristics: length,  $L = 20.0$  m; Young modulus,  $E = 2.1 \times 10^{11}$  N/m<sup>2</sup>; section moment of inertia,  $I = 2.8 \times 10^8$  m<sup>4</sup>; density,  $\rho = 7850.0$  kg/m<sup>3</sup>; stiffness at both ends:  $k_1 = k_2 = 100EI$ . For the purpose of numerical simulations, the beam was modelled by 18 Hermitian beam elements. For the mass matrix, the consistent formulation was considered. The first four natural frequencies of vibration are calculated to be 1.17 Hz, 4.69 Hz, 10.55 Hz and 18.76 Hz. Viscous damping is assumed with 2% of critical for all modes. The mode shapes are shown in Figure 1, where they are scaled to the maximum component equal to one. The modal mass corresponding to these mode shapes are calculated to be 31400.60, 32376.29, 31397.12 and 32343.25. We have generated nineteen different sample records ( $n_r = 19$ ) using:

- Sampling frequency  $f_s = 50$  Hz. Total duration of signals, 100 seconds ( $N = 5000$ ).
- An i.i.d. Gaussian white noise with variance equal to 100 was used as the measured input load for each setup,  $u_t^{(r)}$ . This input has been applied node 1 in setup 1, to node 2 in setup 3, and so on.

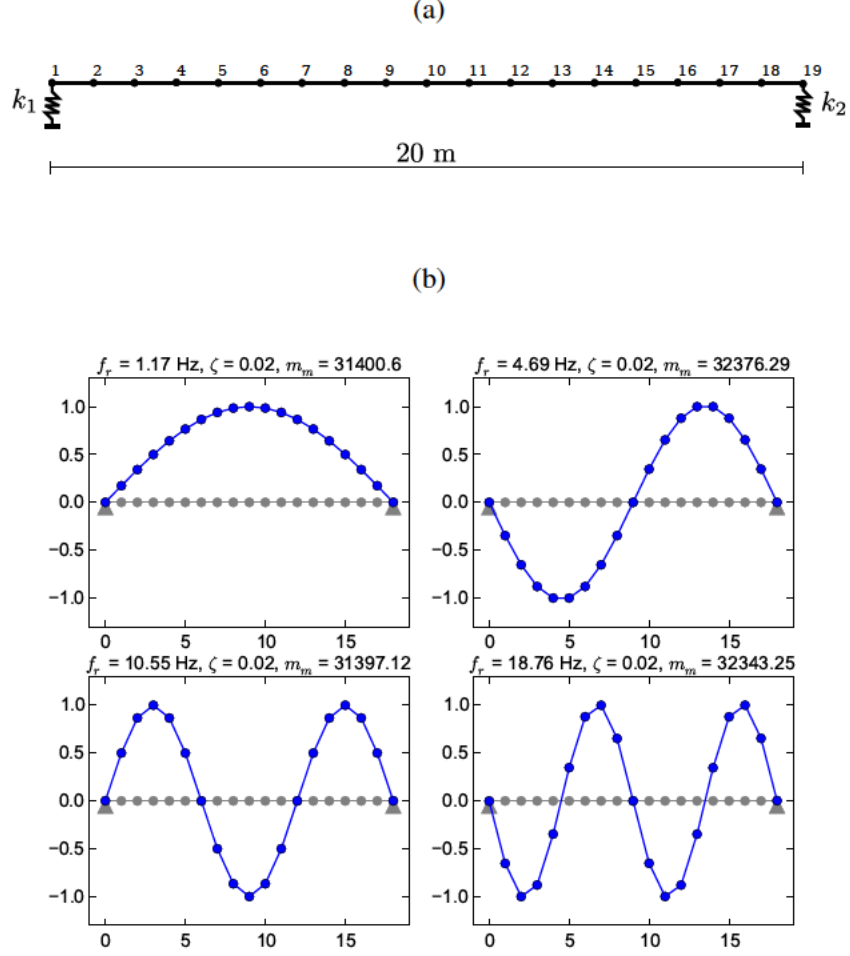


Figure 1: (a) Beam geometry; (b) Modal parameters.

- The beam acceleration was calculated using the mode superposition method taking into account only the first four modes. The vector of measured outputs,  $y_t^{(r)}$ , is composed by the vertical acceleration of nodes 4, 7, 10, 13 and 16.
- Each output was contaminated by i.i.d. Gaussian white noise with variance 0.1 times the variance of the signal.

The state space model (5)-(6) was estimated from the nineteen simulated records using the EM algorithm described in Section 2.2. Then, the modal parameters were computed by mean of equations (18), (19), (20) and (21). The results are showed in Figure 2 and Table 1. We observe that there is good agreement between theoretical parameters and estimated parameters.

Mode	1	2	3	4
MAC	0.9986	0.9999	0.9998	0.9999

Table 1: MAC between real mode shapes and estimated mode shapes.

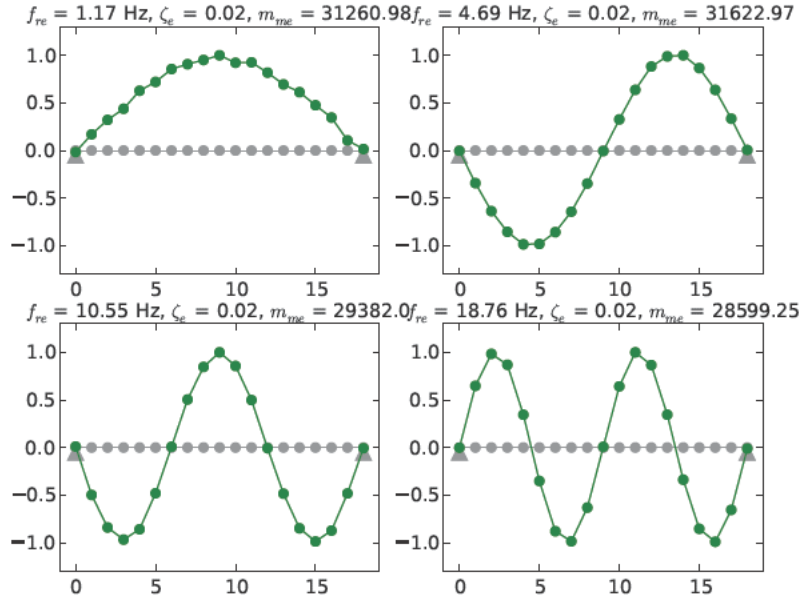


Figure 2: Estimated parameters ( $\square_e$  stands for *estimated*).



Figure 3: Building at the Universidad Politecnica de Madrid.

## 4.2 Experimental data

In this section we use measured data from a building placed at the Universidad Politecnica de Madrid (see Figure 3 for pictures). The building geometry is outlined in Figure 4. The ground floor is devoted to car parking. In the first floor we can distinguish two parts: the first one contains offices and has two stories; the second one is a big room with 12 meters wide, 22 meters long (29 meters if we take into account the angled end) and 6 meters high. With respect to the structure, the beams and columns are made of steel, and the

floors and walls are made of concrete.

On May 16, 2016, researchers from the Universidad Politecnica Madrid conducted some dynamic tests to investigate the modal parameters of the building. In these tests, ten different DOFs were recorded, all vertical accelerations (see Figure 4(a)). Apart from the accelerometers, a 186 N shaker acting in the vertical direction was applied to 27 different points (see Figure 4(b)). For each test the acceleration of the ten accelerometers and the input due to the shaker were recorded at sampling frequency equal to 2048 Hz. Since the modes of interest are in the range 0-15 Hz, the data were re-sampled with new sampling frequency equal to 32 Hz (we used the Matlab function *decimate*).

This data were analyzed using the method proposed in this paper and the results are shown in Figure 5. On the other hand, this building was studied from an operational perspective in [4], so it can be used to compared the results obtained in this study

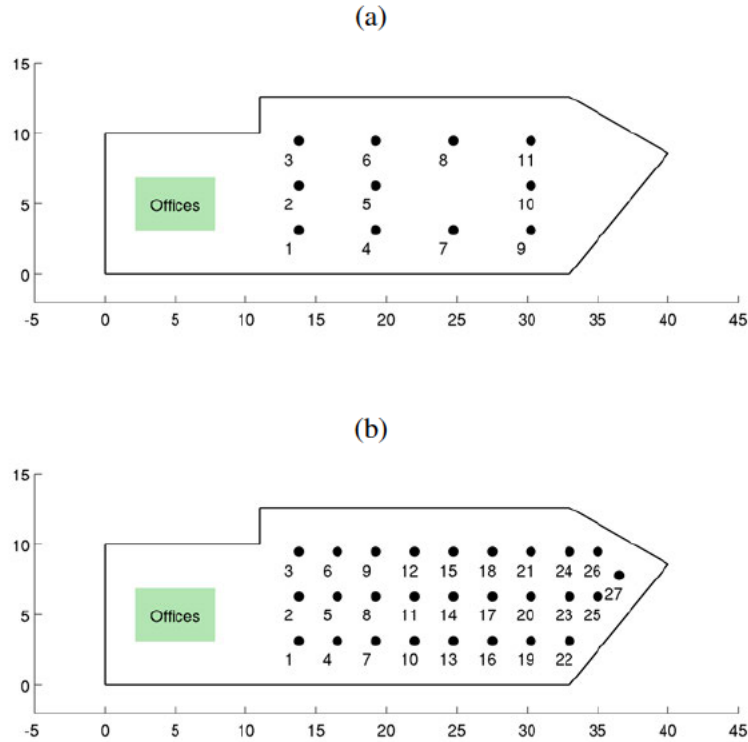


Figure 4: (a) Sensor layout; (b) Shaker layout.

## 5 Conclusions

In this work we have presented a method to deal with roving input tests in the time domain. The method has two parts: the model and its estimation. The model is based on the well known state space model because this model allow us to process multiple sets of records. The estimation of this model was done using the Expectation-Maximization or EM algorithm, which is a maximum likelihood algorithm. A synthetic experiment has been conducted in order to check the effectiveness of the method. Finally, experimental data have been analyzed to prove the feasibility of the method.



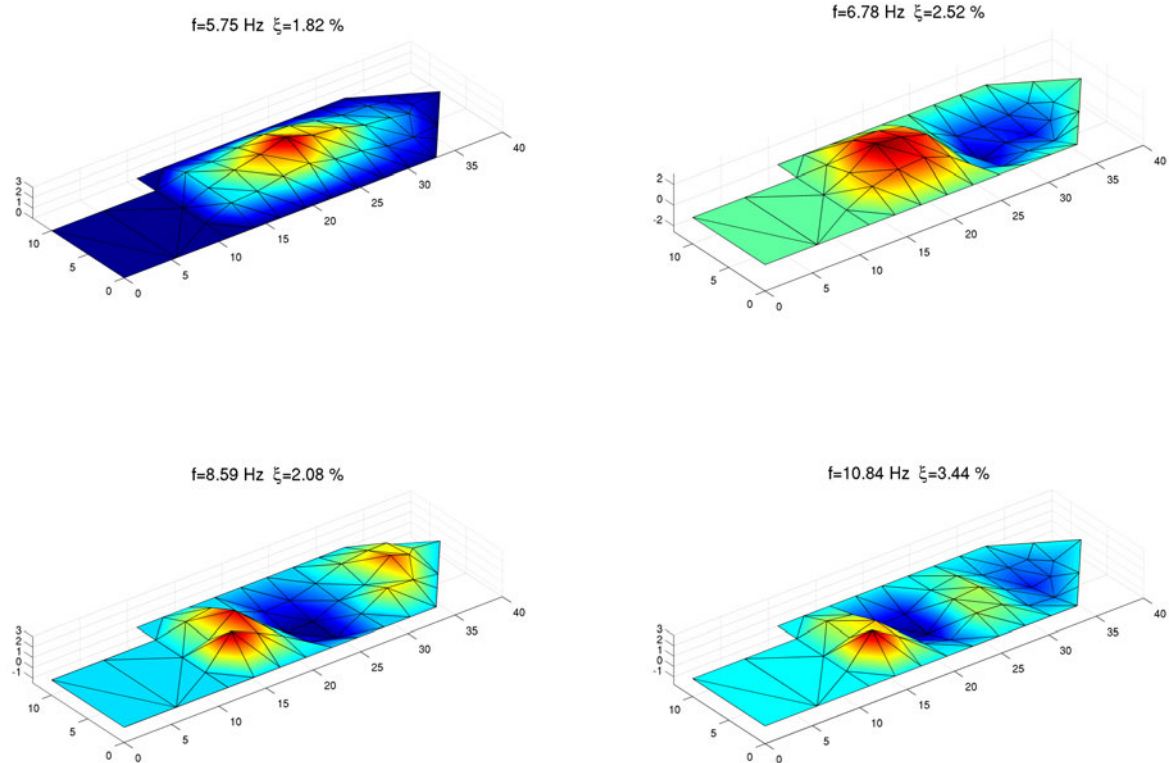


Figure 5: Some estimated mode shapes.

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